Exercise 1

Recall that Q satisfies $\Delta Q - Q + Q^p = 0$ and the operator $\mathcal{L}$ works on h

$$\mathcal{L}h = -i\big[(-\Delta + 1 - Q^{p-1})h - \frac{p-1}{2}Q^{p-1}(h+\bar{h})\big],$$

which can be rewritten into a matrix operator acting on $\begin{pmatrix}\operatorname{Re} h \\ \operatorname{Im} h\end{pmatrix}$:

$$\mathcal{L} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}, \quad L_+ = -\Delta + 1 - pQ^{p-1}, \quad L_- = -\Delta + 1 - Q^{p-1}. \tag{1}$$

Let $1 < p < 2^* - 1$, the functional

$$J[u] = \frac{\|\nabla u\|_{L^2}^{2a}\|u\|_{L^2}^{2b}}{\|u\|_{L^{p+1}}^{p+1}}, \quad a = \frac{d(p-1)}{4}, \quad b = \frac{d+2-(d-2)p}{4}, \quad u \in H^1(\mathbb{R}^d)$$

attains its minimum at $\frac{\psi}{\|\psi\|_{L^2}}$ where

$$\psi = b^{\frac{1}{p-1}}Q(\sqrt{\frac{b}{a}}x).$$

Show that the minimisation inequality

$$\frac{d^2}{d\varepsilon^2}|_{\varepsilon=0}J(\psi+\varepsilon\eta) \geq 0, \quad \forall \eta \in C_0^\infty(\mathbb{R}^d)$$

reads as

$$c(L_+\eta, \eta) \geq \frac{1}{a}\Big(\int \eta\Delta Q\Big)^2 + \frac{1}{b}\Big(\int \eta Q\Big)^2 - \Big(\int \eta Q^p\Big)^2,$$

where $c = \frac{1}{p+1}\int Q^{p+1} = \frac{1}{2a}\int |\nabla Q|^2 = \frac{1}{2b}\int |Q|^2 > 0$.

Hint: Notice that since $(\frac{f}{g})' = 0$, $(\frac{f}{g})'' = \frac{f''}{g} - \frac{fg''}{g^2}$ and hence $(\frac{f}{g})'' \geq 0$ is equivalent to say $f''g - fg'' \geq 0$.

Exercise 2

Recall Q satisfies $\Delta Q - Q + Q^p = 0$ and $L_+ = -\Delta + 1 - pQ^{p-1}$. We consider the minimizing problem

$$\tau := \inf_{\mathcal{A}_+}(L_+f, f),$$
$$\mathcal{A}_+ = \{f \in H^1 \,|\, \|f\|_{L^2} = 1,\, (f, Q) = 0,\, (f, xQ) = 0\}.$$

The minimum is attained at function $f_* \neq 0$, $\|f_*\|_{L^2} = 1$ and $(L_+f_*, f_*) = 0 = \tau$. Show that there exists $(f_*, \lambda, \beta, \gamma)$ among the critical points of the Lagrange multiplier problem

$$(L_+ - \lambda)f = \beta Q + \gamma \cdot xQ, \quad \|f\|_{L^2} = 1, \quad (f, Q) = (f, x_jQ) = 0.$$

Exercise 3

Let $1 < p \leq 1 + \frac{4}{d}$ and $h_0 \in \mathcal{H}$, $r(t) \in \mathcal{H}$. Consider the linear system

$$\partial_t h = \mathcal{L}h + r, \quad h|_{t=0} = h_0.$$

Let $h = f + ig \in \mathcal{H}$.

Show that:

- The linear flow maps from $\mathcal{H}$ to $\mathcal{H}$.
- If $r(t) = 0$, the quantity $(L_+ f, f) + (L_- g, g)$ is conserved by the homogeneous flow.

Exercise 4

Recall the NonLinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = -|u|^{p-1}u, \\ u|_{t=0} = u_0(x), \end{cases} \tag{NLS}$$

and Q is the fundamental solution of $\Delta Q - Q + Q^p = 0$.

Let $\lambda_0, \gamma_0, \xi_0 = (\xi_0^1, \cdots, \xi_0^d), \theta_0 = (\theta_0^1, \cdots, \theta_0^d)$ be real numbers. Show that the functions

$$\lambda_0^{\frac{2}{p-1}} Q(\lambda_0(\theta(x,t) - \theta_0))e^{i(\xi_0 \cdot (\theta(x,t) - \theta_0) + (\gamma(x,t) - \gamma_0))}$$

form a $(2d+2)$-parameter family of solutions of (NLS), $1 < p < 2^* - 1$, if the following relations hold for $\theta = (\theta^1, \cdots, \theta^d)$ and γ:

$$\frac{\partial \theta^i}{\partial t} = -2\xi_0^i, \quad \frac{\partial \theta^i}{\partial x^j} = \delta_i^j, \quad \frac{\partial \gamma}{\partial t} = \lambda_0^2 + |\xi_0|^2, \quad \frac{\partial \gamma}{\partial x^j} = 0. \tag{2}$$

Exercise 5

Recall the perturbed Cauchy problem of the (NLS):

$$i\partial_t u^\varepsilon + \Delta u^\varepsilon = -|u^\varepsilon|^{p-1}u^\varepsilon, \quad u^\varepsilon|_{t=0} = Q(x) + \varepsilon h_0(x), \quad p < 1 + \frac{4}{d}.$$

We expand u^ε as

$$u^\varepsilon(t,x) = \left(\lambda_0^{\frac{2}{p-1}} Q(\lambda_0(\theta - \theta_0)) + \varepsilon h_1 + \varepsilon^2 h_2 + \cdots\right)e^{i(\xi_0 \cdot (\theta - \theta_0) + (\gamma - \gamma_0))},$$
$$h_1 = h_1(\tau, \Theta), \quad \tau = \int_0^t \lambda_0^2 \, dt', \quad \Theta = \lambda_0(\theta - \theta_0),$$

where θ, γ satisfy (2), $\lambda_0 = \lambda_0(\varepsilon t)$, $\theta_0 = \theta_0(\varepsilon t)$, $\gamma_0 = \gamma_0(\varepsilon t)$, $\xi_0 = \xi_0(\varepsilon t)$ and $h_1 \in \mathcal{H}$ for $t > 0$ and for any $T_0 > 0$,

$$\sup_{0 \leq t \leq T_0/\varepsilon} \|\varepsilon h_1(t)\|_{H^1} = \alpha(\varepsilon), \quad \alpha(\varepsilon) \to 0 \text{ as } \varepsilon \to 0. \tag{3}$$

We substitute the above expansion of u^ε with $h_1 = h + h_0$ into the perturbed Cauchy problem. By comparing the terms of order ε, show that

$$\partial_\tau h = \mathcal{L}h + r, \quad h|_{t=0} = 0, \quad h = (f,g)^t,$$

where

$$r = (a,b)^t + \mathcal{L}h_0, \quad a = -\lambda_0^{\frac{2}{p-1}-3}\dot{\lambda}_0(\frac{2}{p-1}Q + \Theta \cdot \nabla Q) + \lambda_0^{\frac{2}{p-1}-1}\dot{\theta}_0 \cdot \nabla Q,$$
$$b = -\lambda_0^{\frac{2}{p-1}-3}\dot{\xi}_0 \cdot \Theta Q + (\xi_0 \cdot \dot{\theta}_0 + \dot{\gamma}_0)\lambda_0^{\frac{2}{p-1}-2}Q.$$

Exercise 6

Let $1 < p < 1 + \frac{4}{d}$. Recall the Lyapunov functional

$$H(u) = 2E(u) + M(u) = \int_{\mathbb{R}^d} |\nabla u|^2 - \frac{2}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} + \int_{\mathbb{R}^d} |u|^2.$$

Let $u_0 \in H^1$, and $u \in C(\mathbb{R}; H^1)$ is the corresponding global solution of the Cauchy problem of (NLS). We write

$$u(t, x + x(t))e^{i\gamma(t)} = e^{it}(Q(x) + h(t,x)),$$

where Q satisfies $-\Delta Q + Q = Q^p$.

Show that

$$H(u_0) - H(Q) \geq (L_+ \operatorname{Re} h, \operatorname{Re} h) + (L_- \operatorname{Im} h, \operatorname{Im} h) - C(\|h\|_{H^1}^{2+\theta} + \|h\|_{H^1}^6),$$

with $\theta > 0$.

Hint: We use the mass and energy conservation laws to show

$$H(u_0) - H(Q) = H(u(t, \cdot + x(t))e^{i\gamma(t)}) - H(e^{it}Q) = H(e^{it}(Q + h)) - H(e^{it}Q),$$

then use the elliptic equation $-\Delta Q + Q = Q^p$.

Exercise 7

Let $h = f + ig$, $f, g \in \mathbb{R}$ be the minimization of $\|h\|_{H^1}^2$, with the restriction $\|Q + h\|_{L^2} = \|Q\|_{L^2}$, where Q satisfies $-\Delta Q + Q = Q^p$. Show that

$$\int Q^{p-1}\partial_{x_j} Q f \, dx = 0, \quad \int Q^p g \, dx = 0.$$